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SOME ASPECTS OF WHITE NOISE QUADRATIC FUNCTIONALS*

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Abstract

We are interested in **quadratic white noise functionals**. First we define the two subspaces $H_2^{(-2,1)}$ and $H_2^{(-2,2)}$ of the space of quadratic functionals of white noise. By introducing a random measure dz(ω), we consider a Hilbert space spanned by $\left\{\int f(u)\ d\,z(u)\right\}$. Then we can see the duality between the two spaces $H_2^{(-2,1)}$ and $H_2^{(-2,2)}$ In addition, when we consider the **Lévy Laplacian** on $H_2^{(-2,1)}$ we can obtain the adjoint of the Lévy Laplacian on $H_2^{(-2,2)}$

Keywords : quadratic white noise functional, Lévy Laplacian

Introduction

We are interested in quadratic functionals of white noise. Before discussing the quadratic functionals of white noise, we first recall the linear functional of white noise.

In the linear case, $(\dot{B})$ can be expressed as Wiener integral of the form

$$\dot{B}(f) = \int f(t)\,\dot{B}(t)\,dt, \quad f \in L^2(R^1)$$

The collection of such $\dot{B}(f)$ spans a Hilbert space H_1 under the $L^2(\Omega, P)$-topology. The above formula proves that the space H_1 is isomorphic to $L^2(R^1)$:

$$H_1 \cong L^2(R^1). \tag{1.1}$$

Extending the space $L^2(R^1)$ to Sobolev space K^{-1}, we obtain the space $H_1^{(-1)}$ of linear functional of white noise which is the corresponding extension of H_1.

Namely, we have a generalization of (1.1) :

$$H_1^{(-1)} \cong K^{-1}(R^1). \tag{1.2}$$

In reality, $H_1^{(-1)}$ is defined by (1.2). The space $H_1^{(-1)}$ is the space of linear functionals of white noise.

This isomorphism (1.2) determines the $H_1^{(-1)}$-norm denoted by $\|\ \|_{-1}$. More precisely, if $(\in H_1^{(-1)})$ corresponds to $f\ (\in K^{-1})$ by (1.2) then $\|\varphi\|_{-1} = \|f\|_{K^{-1}(R^1)}$.

It is not difficult to deal with linear functionals of $\dot{B}(t)$'s, but when we deal with nonlinear functionals of $\dot{B}(t)$'s, we need to renormalize them. We can see the idea of the renormalization by explaining for the quadratic forms of $\dot{B}(t)$'s.

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2. Quadratic functionals of white noise (Ref.[4])

We now consider the quadratic functionals which is actually nonlinear functionals of $\dot{B}(t)$'s. We start by giving the definition of quadratic functionals.

2.1 Definition Let $Q(x) = Q(x(t))$, $t \in R^1$ be an $(L^2)^-$ *-functional and its S-transform be* $U(\xi)$. For any $\xi, \eta \in E$ and for any $\alpha, \beta \in R^1$, the function $U(\alpha\xi+\beta\eta)$ is a *homogeneous polynomial of degree 2 in* α *and* β, then $U(\xi)$ and hence $Q(x)$ is called a **quadratic functional or quadratic form**.

If the function $U(\alpha\xi+\beta\eta)$ is a *homogeneous polynomial of degree n in* α *and* β, then U and Q are called **entire homogeneous of degree n**.

Among others, the subspace $H_2^{(-2)}$ of $(L^2)^-$ consisting of *quadratic generalized white noise functionals* is particularly important.

Before we discuss the details of the analysis of quadratic forms of the $\dot{B}(t)$, we shall emphasize the significance of "quadratic".

Let $\{X_n\}$ be the independent identically distributed sequence such that each X_n being $N(0, 1)$ random variable. We can express the quadratic functional as

$$Q(X) = \sum_{1\le i,\, j\le n} a_{ij} X_i X_j = Q_1(X) + Q_2(X), \tag{2.1}$$

where

$$Q_1(X) = \sum_{i=1}^{n} a_{ii} X_i^2 \tag{2.2}$$

$$Q_2(X) = \sum_{1\le i,\, j\le n,\, i\ne j} a_{ij} X_i X_j, \quad a_{ij} = a_{ji}, \tag{2.3}$$

We consider the condition for convergence is quasi convergence as $n \to \infty$. Thus, we require the convergence for $\sum a_{ii}$ and square summability of coefficients a_{ij} and also, quasi-convergence of $Q_2(X)$ as $n \to \infty$.

We are now ready to discuss *the passage from discrete to continuous* (in Hida's words the passage from digital to analogue).

Thus we take X_j^n to be the variation of a Brownian motion with the idea of Lévy's construction of Brownian motion by interpolation. Namely, we set $X_j^n = \frac{\Delta_j^n B}{\sqrt{\Delta^n}}$, where $\{\Delta_j^n\}$ is the partition of [0, 1] with $\Delta_j^n = \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right]$ i.e. $\bigcup_{j=1}^{n} \Delta_j^n = [0,1]$ and we can write Δ^n instead of Δ_j^n. Hence $\left|\Delta_j^n\right| = \left|\Delta^n\right| = \frac{1}{2^n}$.

Then ,

$$Q_2(X) = \sum_{i \neq j} a^n_{ij} \frac{\Delta^n_i B}{\sqrt{|\Delta^n|}} \frac{\Delta^n_j B}{\sqrt{|\Delta^n|}}$$

converges with the assumption

$$\sum \left(a^n_{ij}\right)^2 < \infty, \tag{2.4}$$

as n $\to\infty$.

The above inequality (2.4) guarantees the convergence of

$$\sum_{i \neq j} \frac{a^n_{ij}}{|\Delta^n|} \frac{\Delta^n_i B}{|\Delta^n|} \frac{\Delta^n_j B}{|\Delta^n|} \left|\Delta^n\right| \left|\Delta^n\right|,$$

as letting $|\Delta^n| \to 0$ as n $\to\infty$. The sum tends to

$$Q_2(\dot{B}) = \int_0^1 \int_0^1 F(\mathrm{u}, \mathrm{v})\, \dot{B}(\mathrm{u}) \dot{B}(\mathrm{v})\, \mathrm{du\, dv} \tag{2.5}$$

in H_2, where $\frac{a^n_{ij}}{\Delta^n}$ approximates F(u, v) $\in L^2\widehat{(R^2)}$ so that

As for the limit of $Q_1(X)$, we have to consider as follows. First we note that as $\Delta \to \{t\}$ the quantity $\left(\frac{\Delta B}{\Delta}\right)^2$ may be considered to tend to $\dot{B}(t)^2$, however it is not a generalized white noise functional. If we have the difference $\left(\frac{\Delta B}{\Delta}\right)^2 - \frac{1}{\Delta}$, then it converges to a generalized quadratic functional which is denoted by $:\dot{B}(t)^2:$.

We must modified $Q_1(X)$

$$Q_1(X) \to Q_1{}'(X) = \sum_i a^n_{ii} \left(\frac{\Delta^n_i B}{|\Delta^n|}\right)^2 - \frac{1}{|\Delta^n|}$$

which tends to

$$Q_1(\dot{B}) = \int_0^1 f(\mathrm{u}) :\dot{B}(u)^2: \mathrm{du}, \tag{2.6}$$

with $f \in L^2(R^1)$.

From the above modification, we can see the idea of renormalization. The result (2.6) is a Hida distribution while (2.5) defines an ordinary H_2-functional.

So far we observed some particular generalized functionals $\dot{B}(t)$ of degree 2, where we have seen the idea of the passage from discrete to continuous. The limit of $Q(X)$, is in fact, ***normal functional*** in terms of P. Lévy,

$$\int f(\mathrm{u}) :\dot{B}(u)^2: \mathrm{du} + \int\int F(\mathrm{u}, \mathrm{v}) :\dot{B}(\mathrm{u}) \dot{B}(\mathrm{v}): \mathrm{du\, dv}. \tag{2.7}$$

We just mentioned above that equation (2.7) is the normal functionals in $\dot{B}(t)$. Thus, we should now give the general expression of *Lévy's normal functional* in terms of *S*-transform which is of the form

$$\int f(\mathrm{u})\,\xi(\mathrm{u})^2 \mathrm{du} + \int\int F(\mathrm{u},\mathrm{v})\,\xi(\mathrm{u})\,\xi(\mathrm{v})\mathrm{du}\,\mathrm{dv}, \tag{2.8}$$

where $\boldsymbol{f} \in L^1(R)$, $F \in L^2(R^2)$ and $\xi \in \mathrm{E}$.

We now pause to explain the importance of normal functionals. Most significant reason is in the analytic property.

We may say that the first term of (2.8) has the Fréchet derivative and the kernel function has singularity only on the diagonal. We may therefore call it the singular part of the normal functional, while the second term is second Fréchet differentiable with kernel function F(u, v). The following examples are given to see the singular and regular parts.

2.2 Example Let

$$U(\xi) = \int f(\mathrm{u})\,\xi(\mathrm{u})^2 \mathrm{du}.$$

Taking its variation

$$\delta U(\xi) = 2\int f(\mathrm{u})\,\xi(\mathrm{u})\,\delta\,\xi(\mathrm{u})\,\mathrm{du}$$

$$U'(\xi, t) = 2\, f(t)\,\xi(t) = 2\int f(\mathrm{u})\,\xi(\mathrm{u})\,\delta_{\mathrm{t}}(\mathrm{u})\,\mathrm{du}$$

$$\delta U'(\xi, t) = 2\int f(\mathrm{u})\,\delta\xi(\mathrm{u})\,\delta_{\mathrm{t}}(\mathrm{u})\,\mathrm{du}$$

$$U''(\xi, t) = 2\, f(t)\,\frac{1}{\mathrm{dt}}$$

Thus we can see that second *Fréchet derivative* does not exist. It is the singular part of the normal functional (2.8).

2.3 Example Let

$$U(\xi) = \int_{R^2} F(\mathrm{u},\mathrm{v})\xi(\mathrm{u})\xi(\mathrm{v})\mathrm{du}\,\mathrm{dv}$$

with smooth symmetric kernel F(u, v).

The variation of U is

$$\delta U = 2\int F(\mathrm{t},\mathrm{u})\,\xi(\mathrm{u})\,\delta\,\xi(\mathrm{u})\,\mathrm{du},$$

thus

$$U'(\xi) = 2\int F(\mathrm{t},\mathrm{u})\,\xi(\mathrm{u})\,\mathrm{du}$$

$$U''(\xi) = 2\,F(t,t)\;\xi(t).$$

We now see that it is second Fréchet differentiable and so it is the regular part of the normal functional (2.8).

Some other characteristics of the quadratic functional can be seen through the representation by using the T or S transform.

For an ordinary functionals, we are given a representation in terms of symmetric $L^2(R^2)$ function. We can then appeal to Mercer's theorem to develop the original quadratic random functional decomposed into the sum of countably many independent random variables with χ^2 – distribution.

A plausibility of renormalization can be seen from the Quadratic forms of white noise functionals. Reductionism, which is Hida's favorite idea, suggests us to start the analysis with the algebra of polynomials in $\dot{B}(t)$'s.

Except linear functionals, those polynomials need renormalization in order to be generalized functionals of $\dot{B}(t)$'s. Quadratic form that is the limit of $Q_1(x)$ shows why and how the renormalization is necessary to be a generalized quadratic functional. We emphasize that such a generalized functional plays significant role also in the applications in physics.

When Laplacian is discussed, normal quadratic forms illustrate how it works from the view point of "harmonic analysis", where infinite dimensional rotations are acting.

Remark. The harmonic property can be discussed after the Lévy Laplacian Δ_L is defined. Both Q_1 and Q_2 have enough analytic properties, in particular they are in the domain of Δ_L. We see, as mentioned before, that Q_2 is always harmonic, while Q_1 may have non-zero trace.

3. Space of quadratic functionals of white noise (Ref. [4])

Let $K^{-3/2}(R^2)$ be the dual space of Sobolev space $K^{3/2}(R^2)$. There is an isomorphism

$$H_2^{(-2)} \cong \widehat{K^{-3/2}(R^2)} \tag{3.1}$$

as an extension of the known isomorphism

$$H_2 \cong \widehat{L^2(R^2)}. \tag{3.2}$$

We now have the Gel' fand triple

$$\mathrm{H}_2^{(2)} \subset \mathrm{H}_2 \subset \mathrm{H}_2^{(-2)}, \tag{3.3}$$

where $\mathrm{H}_2^{(2)}$ is the dual space of $\mathrm{H}_2^{(-2)}$, defined in the usual manner based on the scalar product in H_2.

According to the isomorphism (3.1), for $\varphi \in \mathrm{H}_2^{(-2)}$ there is a function $F(\mathrm{u}, \mathrm{v})$ in the space $\widehat{K^{-3/2}(R^2)}$ to have the representation

$$\varphi(\dot{B}) = \int F(\mathrm{u},\mathrm{v}) : \dot{B}(\mathrm{u})\dot{B}(\mathrm{v}) : \mathrm{du\,dv}. \tag{3.4}$$

where the notation $: - :$ may be considered as the Wick product, i.e. renormalized product.

In the above we have obtained the Gel'fand triple

$$H_2^{(2)} \subset H_2 \subset H_2^{(-2)}, \tag{3.5}$$

where

$$H_2^{(2)} = \{\int\int_{I^2} F(u,v) : \dot{B}(u)\dot{B}(v) : du\, dv\}, F \in \widehat{K^{3/2}(I^2)} \tag{3.6}$$

$$H_2 = \{\int\int_{I^2} F(u,v) : \dot{B}(u)\dot{B}(v) : du\, dv\}, F \in \widehat{K^{2}(I^2)} \tag{3.7}$$

and the quadratic Hida distribution space

$$H_2^{(-2)} = \{\int\int_{t^2} F(u,v) : \dot{B}(u)\dot{B}(v) : du\, dv\}, F \in \widehat{K^{3/2}(I^2)}, \tag{3.8}$$

where $\hat{}$ means symmetric.

It can be seen that the space $H_2^{(-2)}$ is the space of quadratic functionals of $\dot{B}(t)$, $t \in I = [0, 1]$. We define the new subspace of $H_2^{(-2)}$:

$$H_2^{(-2,1)} = \{ \int_I f(u) : \dot{B}(u)^2 : du, f \in L^2(I)\} \tag{3.9}$$

The function f is viewed as $f(\frac{u+v}{2})\, \delta(u - v) \equiv f(u)$.

$$(\int_I f(u) : \dot{B}(u)^2 : du, \int_I g(u) : \dot{B}(u)^2 : du) = (f, g)_{K^{-3/2}} \tag{3.10}$$

The null space of $H_2^{(-2,\,1)}$ is {0}. Hence the Hilbert space $H_2^{(-2,\,1)}$ is defined as a subspace of $H_2^{(-2)}$. We now introduce a new vector space in a formal expression

$$H_2^{(-2,\,2)} = \{ \int_I g(u) : \dot{B}(u)^2 : du^2, g \in L^2(I)\}. \tag{3.11}$$

Here we give the interpretation of the elements in $H_2^{(-2,\,2)}$ as follows.

$$\left|\Delta_{2k}^{(n+1)}\right| = \left|\Delta_{2k+1}^{(n+1)}\right| = 2^{-(n+1)} \left(= \Delta^{(n+1)}\right),$$

We consider the following conditional expectation.

$$E\left(: \left(\frac{\Delta_{2k}^{(n+1)} B}{\sqrt{\Delta^{(n+1)}}}\right)^2 : + : \left(\frac{\Delta_{2k+1}^{(n+1)} B}{\sqrt{\Delta^{(n+1)}}}\right)^2 : \middle| : \left(\frac{\Delta_k^{(n)} B}{\sqrt{\Delta^{(n)}}}\right)^2 : \right) = : \left(\frac{\Delta_k^{(n)} B}{\sqrt{\Delta^{(n)}}}\right)^2 :$$

Then we can prove that

$$E\left[\sum_k \left(: \left(\frac{\Delta_{2k}^{(n+1)} B}{\sqrt{\Delta^{(n+1)}}}\right)^2 : + : \left(\frac{\Delta_{2k+1}^{(n+1)} B}{\sqrt{\Delta^{(n+1)}}}\right)^2 : \middle| : \frac{\left(\Delta_k^{(n)} B\right)^2}{\Delta^{(n)}} :, 0 \le k \le 2^n \right)\right]$$

$$= \sum_k E\left[\left(:\left(\frac{\Delta_{2k}^{(n+1)}B}{\sqrt{\Delta^{(n+1)}}}\right)^2 : + :\left(\frac{\Delta_{2k+1}^{(n+1)}B}{\sqrt{\Delta^{(n+1)}}}\right)^2 : \middle| \ :\frac{\left(\Delta_k^{(n)}B\right)^2}{\Delta^{(n)}}: \right)\right]$$

$$= \sum_k :\left(\frac{\Delta_k^{(n)}B}{\sqrt{\Delta^{(n)}}}\right)^2 :$$

Hence

$$E\left[\sum_k :\frac{\left(\Delta_k^{(n+1)}B\right)^2}{\Delta^{(n+1)}} : \Delta^{(n+1)} \middle| \ :\left(\Delta_k^{(n)}B\right)^2 :, 0 \le k \le 2^n - 1\right]$$

$$= \sum_k :\left(\frac{\Delta_k^{(n)}B}{\Delta^{(n)}}\right)^2 : \Delta^{(n)}$$

Thus, for the quadratic form $\sum_j :\left(\frac{\Delta_j^{(n)}B}{\Delta^{(n)}}\right)^2 : \Delta^{(n)}$, we obtain its average

$$\frac{1}{2^n}\sum_j :\left(\frac{\Delta_j^{(n)}B}{\Delta^{(n)}}\right)^2 : \Delta^{(n)} = \sum_j :\left(\frac{\Delta_j^{(n)}B}{\Delta^{(n)}}\right)^2 : \left(\Delta^{(n)}\right)^2 \tag{3.12}$$

is consistent in n by the projection which is realized by the conditional expectation $E(\cdot|B_n)$ where B_n is generated by $:\left(\Delta_j^{(n)}B\right)^2 :$, $0 \le j \le 2n - 1$. Thus $\int_I : \dot{B}(t)^2 : (dt)^2$ is its limit.

The topology is to be introduced below so that the space $H_2^{(-2,\ 2)}$ has meaning. Let us set

$$dz(u) = \frac{1}{\sqrt{2}} : \dot{B}(u)^2 : (du)^{\frac{3}{2}} \tag{3.13}$$

Thus, we have a system of independent (hence orthogonal) infinitesimal random variables and $E\,(: \dot{B}(u)^2 :^2\,) = \frac{2}{(du)^2}$.

Thus, $dz(u)$ is a random measure with $E(\ |\ dz(u)\ |^2\ ) = du$ and we can define a Hilbert space L spanned by $\left\{ \int f(u)dz(u) \right\}$.

The bilinear form $\ll\cdot, \cdot\gg$ which connects the two spaces $H_2^{(-2,\ 1)}$ and $H_2^{(-2,\ 2)}$ is given by the Hilbertian norm $\langle\cdot , \cdot\rangle$ with respect to the random measure $dz(u)$ as in the following manner.

For the two functionals $\varphi \in H_2^{(-2,\,1)}$ and in $H_2^{(-2,\,2)}$ such that

$$\varphi = \int f(\mathrm{u}) : \dot{B}(u)^2 : \mathrm{du}$$

and

$$\psi = \int g(\mathrm{u}) : \dot{B}(u)^2 : (\mathrm{du})^2,$$

$$\langle\!\langle \varphi, \psi \rangle\!\rangle = \langle \int f(\mathrm{u}) : \dot{B}(u)^2 : \mathrm{du}, \int g(\mathrm{u}) : \dot{B}(u)^2 : (\mathrm{du})^2 \rangle_{\mu}$$

$$= \langle \int f(\mathrm{u}) : \dot{B}(u)^2 : (\mathrm{du})^{\frac{3}{2}} , \int g(\mathrm{u}) : \dot{B}(u)^2 : (\mathrm{du})^{\frac{3}{2}} \rangle_{\mu}$$

$$= E[(\int f(\mathrm{u}) : \dot{B}(u)^2 : (\mathrm{du})^{\frac{3}{2}})(\int g(\mathrm{u}) : \dot{B}(u)^2 : (\mathrm{du})^{\frac{3}{2}})]$$

$$= \int f(\mathrm{u}) g(\mathrm{u})\, \mathrm{du}$$

since $E\,(: \dot{B}(u)^2 :^2\,) = \frac{2}{(\mathrm{du})^2}$. We denote $\langle\!\langle \varphi, \psi \rangle\!\rangle$ by $f(\varphi)$. Thus f_ψ is a continuous linear functional defined on $H_2^{(-2,\,1)}$.

3.1 Theorem $H_2^{(-2,\,1)}$ and $H_2^{(-2,\,2)}$ are the dual pair with respect to random measure d$z(u)$.

4 The adjoint of the Lévy Laplacian Δ_L

An infinite dimensional Lévy Laplacian (Ref.[1], [3])

$$\lim_{n\to\infty} \frac{1}{n} \sum_{1}^{\infty} \frac{\partial^2}{\partial \xi_i^2}$$

is rephrased in terms of the ∂t (H – H Kuo)

$$\Delta_L = \int \partial_t^2 (\mathrm{dt})^2 \,. \tag{4.1}$$

We define the operator

$$\Delta_L^* = \int (\partial_t^*)^2 (\mathrm{dt})^2 \tag{4.2}$$

in a formal expression. Any member of $H_2^{(-2,\,1)}$ is in the domain of Δ_L.

4.1 Theorem By the above relation with the choice of the dual pair in the previous section, Δ_L^* can be understood as the adjoint of the Lévy Laplacian Δ_L.

Proof. Since any member of $H_2^{(-2,\,1)}$ is in the domain of Δ_L, we take

$$\varphi = \int f(u) : \dot{B}(u) :^2 \mathrm{d}u \in \mathrm{H}_2^{(-2,\,1)}.$$

Take the *S*-transform we obtain the *U*-functional, we have

$$U(\xi) = \int f(u)\,\xi(u)^2\,\mathrm{d}u.$$

In Example 2.2, we have obtained

$$U''(\xi, t) = 2\,f(t)\,\frac{1}{\mathrm{d}t}\,.$$

Taking back S^{-1}-transform

$$\partial_t^2\,\varphi = 2\,f(t)\,\frac{1}{\mathrm{d}t}$$

$$\Delta_L\varphi = \int \partial_t^2\varphi\,(\mathrm{d}t)^2$$

$$= \int 2\,f(t)\,\mathrm{d}t$$

$$= \text{constant}$$

On the other hand we take

$$\psi = \int g(u) : \dot{B}(u)^2 : (\mathrm{d}u)^2 \in \mathrm{H}_2^{(-2,\,2)} \tag{4.3}$$

then

$$\Delta_L^* = \int_I (\partial_t^*)^2 (\mathrm{d}t)^2 \int g(u) : \dot{B}(u)^2 : (\mathrm{d}u)^2 \tag{4.4}$$

Therefore

$$\langle \Delta_L\,\varphi, \psi \rangle = \langle 2\int f(u)\,\mathrm{d}u\,,\ \int g(u) : \dot{B}(u)^2 : (\mathrm{d}u)^2 \rangle_\mu$$

$$= E[\,2\int f(u)\,\mathrm{d}u \int g(u) : \dot{B}(u)^2 : (\mathrm{d}u)^2\,]$$

$$= 2\int f(u)\,\mathrm{d}u \int g(u)\,\frac{1}{\mathrm{d}u}\,(\mathrm{d}u)^2$$

$$= 2\int f(u)\,\mathrm{d}u \int g(u)\,\mathrm{d}u \tag{4.5}$$

On the other hand

$$\langle \varphi,\ \Delta_L^*\,\psi \rangle = \langle \int f(u) : \dot{B}(u)^2 : \mathrm{d}u\,,\ \Delta_L^* \int g(u) : \dot{B}(u)^2 : (\mathrm{d}u)^2 \rangle_\mu$$

$$= E[\int f(u) : \dot{B}(u)^2 : \mathrm{d}u \int (\partial_u^*)^2 \left(\mathrm{d}u\right)^2 \int g(u) : \dot{B}(u)^2 : (\mathrm{d}u)^2\,]$$

$$= E[\int f(u) : \dot{B}(u)^2 : \mathrm{d}u \int : \dot{B}(u)^2 : \left(\mathrm{d}u\right)^2 \int g(u) : \dot{B}(u)^2 : (\mathrm{d}u)^2\,]$$

$$= \int f(u)\,\mathrm{d}u\,\frac{1}{\mathrm{d}u} \int g(u)\,\frac{2}{(\mathrm{d}u)^2}\,(\mathrm{d}u)^4$$

$$= 2\int f(u)\,\mathrm{d}u \int g(u)\,\mathrm{d}u \qquad (4.6)$$

The equations (4.5) and (4.6) gives

$\langle \Delta_L\, \varphi, \psi \rangle = \langle \varphi,\ \Delta_L^{*} \psi \rangle$.

Thus, the assertion is proved.

Conclusion

In this paper we have constructed the dual pair which are the subspaces $H_2^{(-2,\,1)}$ and $H_2^{(-2,\,2)}$ of the space H_2 of quadratic white noise functionals. Consequently the adjoint of Lévy Laplacian can also be constructed.

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